U. S. AIR FORCE

PROJECT RAND

RESEARCH MEMORANDUM

MORE ON GAMES OF SURVIVAL

M. P. Peisakoff

RM-884

20 June 1952

Assigned to ______________________

This is a working paper. It may be expanded, modified, or withdrawn at any time.

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **20 JUN 1952**	2. REPORT TYPE	3. DATES COVERED **00-00-1952 to 00-00-1952**

4. TITLE AND SUBTITLE **More on Games of Survival**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Rand Corporation,Project Air Force,1776 Main Street, PO Box 2138,Santa Monica,CA,90407-2138**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**22**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

SUMMARY. Let Ω be the game of survival which is the repetition (until the bankruptcy of one of the players) of a normalized finite zero-sum two-person game, $\Gamma = (\Gamma_{ij})$, where each Γ_{ij} is a non-zero integer. It is shown that Ω is inessential and has some easily described optimal strategies. It is also shown that if $\max_{i,j} |\Gamma_{ij}|$ is small enough compared to the combined fortunes, then playing at the n-th play a δ^n-optimal strategy for Γ is an ε-optimal strategy for Ω, if δ is sufficiently small.

MORE ON GAMES OF SURVIVAL

M. P. Peisakoff

We are interested in the games $\{\Omega(f_1, f_2)\}$ in which two players with finite fortunes, f_1 and f_2, respectively, in chips repeat a normalized finite zero-sum two-person game, $\Gamma = (\Gamma_{ij} | (1, 1) \leq (i, j) \leq (i_0, j_0))$. At least one of f_1 and f_2 is positive and play is continued until the fortune of one of the players is non-positive, or ad infinitum if this never occurs. The payoff in money is (1, 0) if player 2 ends with a non-positive fortune and (0, 1) if player 1 ends with a non-positive fortune. If the game goes on indefinitely, then the payoff is $(\alpha(C), \beta(C))$, which can depend on the course of the game, C, but which satisfies $(\alpha(C), \beta(C)) \leq (1, 1)$ and $\alpha(C) + \beta(C) \leq 1$. We shall show that if all the Γ_{ij}'s are non-zero integers, then Ω is inessential* and has some easily described optimal strategies. (In an inessential game, an optimal strategy for a player is one which secures for the player the maximum amount he can insure for himself. An ε-optimal strategy secures for

*An inessential game is one in which the players together can secure only the sum of the (minorant) amounts insurable without cooperation.

him at least that amount less ε.) We shall alsoshow that if $\max_{i,j} |\Gamma_{ij}|$ is small enough compared to the combined fortunes, then playing at the n-th play a δ^n-optimal strategy for Γ is an ε-optimal strategy for Ω, if δ is sufficiently small. (δ^n is the n-th power of δ.)

We assume that every column of Γ has a positive entry, and every row has a negative entry. Otherwise, there would be a negative column or a positive row. In the first case, player 2 can always force player 1's fortune to become non-positive, by playing the negative column repeatedly. In the second case, player 1 can force player 2's fortune to become non-positive by playing the positive row repeatedly.

Let $\bar{\Omega}^{(n)}(f_1, f_2)$ be the game in which the two players repeat Γ n times, or until one of the players has a non-positive fortune if this occurs first. The payoff in money is (0, 1) if player 1 ends with a non-positive fortune, and (1, 0) otherwise. $\bar{\Omega}^{(n)}(f_1, f_2)$ is a constant-sum two-person game with value, say, $(\bar{v}^{(n)}(f_1, f_2), 1 - \bar{v}^{(n)}(f_1, f_2))$. We observe

(1) Player 2 can always win as much money in $\bar{\Gamma}^{(n+1)}$ as in $\bar{\Gamma}^{(n)}$ by playing a $\bar{\Gamma}^{(n)}$-optimal strategy during the first n moves of $\bar{\Gamma}^{(n+1)}$ and arbitrarily on the (n + 1)th move. Hence

$$\bar{v}^{(n)}(f_1, f_2) \geq \bar{v}^{(n+1)}(f_1, f_2) .$$

(2) Since each column has a positive entry, by repeatedly playing the strategy which assigns each pure strategy probability $1/i_0$, player 1 insures that no matter what player 2 does, player 2's fortune will decrease each time with probability

at least $1/i_0$. Player 1 thereby insures that with probability at least $i_0^{-[f_2]-1}$, player 2 will be bankrupted in at most $[f_2] + 1$ trials. ($[f_2]$ is the largest integer not larger than f_2.) Hence if $n \geqq [f_2] + 1$ and $(f_1, f_2) > (0, 0)$, then

$$\bar{v}^{(n)}(f_1, f_2) \in [\delta, 1]$$

where $\delta = i_0^{-[f_2]-1}$. By definition we have also

$$\bar{v}^{(n)}(f_1, f_2) = 0 \quad \text{if} \quad f_1 \leqq 0$$

and

$$\bar{v}^{(n)}(f_1, f_2) = 1 \quad \text{if} \quad f_2 \leqq 0 \ .$$

(3) Let $G(\Delta)$ be the game value of Δ, for each game Δ. If $(f_1, f_2) > 0$, after one move of $\bar{\Gamma}^{(n+1)}(f_1, f_2)$, the players are playing $\bar{\Gamma}^{(n)}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})$. Hence

$$\bar{v}^{(n+1)}(f_1, f_2) = G\left(\bar{v}^{(n)}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})\right) \ .$$

(4) Let $\varepsilon \geqq 0$. Player 1 can always win as much in $\bar{\Gamma}^{(n)}(f_1 + \varepsilon, f_2 - \varepsilon)$ as in $\bar{\Gamma}^{(n)}(f_1, f_2)$. Hence

$$\bar{v}^{(n)}(f_1 + \varepsilon, f_2 - \varepsilon) \geqq \bar{v}^{(n)}(f_1, f_2) \ .$$

We can now conclude:

$(\bar{A})$ From (1) and (2),

$$\bar{v}^{(n)}(f_1, f_2) \longrightarrow \bar{v}(f_1, f_2) \begin{cases} \in [\delta, 1] & \text{if } (f_1, f_2) > (0, 0) \\ = 0 & \text{if } f_1 \leqq 0 \\ = 1 & \text{if } f_2 \leqq 0 . \end{cases}$$

($\bar{B}$) From (3), if $(f_1, f_2) > (0, 0)$,

$$\bar{v}(f_1, f_2) = G\left(\bar{v}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})\right) .$$

($\bar{C}$) From (4), for $\varepsilon \geqq 0$,

$$\bar{v}(f_1 + \varepsilon, f_2 - \varepsilon) \geqq \bar{v}(f_1, f_2) .$$

Definition: _A strategy for player 1 is called_ conditionally optimal _if the conditional distribution of his strategy at any play of_ Γ, _given the course of the game up to that play, is an optimal strategy for the game_ $\left(v(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})\right)$ _where_ (ϕ_1, ϕ_2) _is the fortune distribution immediately before the play in question._

Lemma $\bar{1}$: _If player 1's strategy is_ conditionally optimal, _and if with probability one the fortune of one of the players (not necessarily always the same one) eventually becomes non-positive, then player 1 can expect at least_ $\bar{v}(f_1, f_2)$ _in payoff._

Proof: It is sufficient to show that the probability that player 2's fortune becomes non-positive is at least $\bar{v}(f_1, f_2)$. Let $\left((F_1^n, F_2^n) | n \geqq 1\right)$ be the random variable of fortunes at play n, where if the game ends at play N, $(F_1^{N+j}, F_2^{N+j}) \equiv (F_1^N, F_2^N)$ for $j \geqq 1$. Then since player 1's

strategy is conditionally optimal, if $(F_1^n, F_2^n) > (0, 0)$,

$$E\bar{v}(F_1^{n+1}, F_2^{n+1}) \geqq EG\big(\bar{v}(F_1^n + \Gamma_{ij}, F_2^n - \Gamma_{ij})\big)$$

$$= E\bar{v}(F_1^n, F_2^n) ,$$

while

$$E\bar{v}(F_1^{n+1}, F_2^{n+1}) = E\bar{v}(F_1^n, F_2^n)$$

otherwise. Hence, by induction,

$$E\bar{v}(F_1^n, F_2^n) \geqq \bar{v}(f_1, f_2) .$$

Let $\big((P_1^n, P_2^n) | n \geqq 1\big)$ be the random variable which is

(0, 0) if neither player's fortune is non-positive by the end of the n-th play,

(0, 1) if the first player's fortune is non-positive by the end of the n-th play,

(1, 0) if the second player's fortune is non-positive by the end of the n-th play.

Then

$$E\bar{v}(F_1^n, F_2^n) \leqq EP_1^n + E(1 - P_2^n - P_1^n) .$$

But by assumption the second term on the right tends to zero. Hence, where $\varepsilon_n \longrightarrow 0$,

$$EP_1^n + \varepsilon_n \geqq v(f_1, f_2) ,$$

which is the desired result.

Lemma $\bar{2}$: <u>There is a conditionally optimal strategy for the first player which insures that the probability that the game ends by the n-th play tends to one as n tends to ∞, uniformly in the opponent's strategy.</u>

Proof: First we show that for each $(\phi_1, \phi_2) > (0, 0)$, there is an optimal strategy I for the first player for the game $\left(\bar{v}(\phi_1 + \Gamma_{ij}, \phi_1 - \Gamma_{ij})\right)$ such that for all J, $\Pr\left\{\Gamma_{IJ} > 0\right\} > 0$. Suppose, to the contrary, that for some $(\phi_1, \phi_2) > (0, 0)$, for all optimal I, there is a J such that $\Pr\left\{\Gamma_{IJ} > 0\right\} = 0$, or what is the same since $\Gamma_{ij} \neq 0$, $\Pr\left\{\Gamma_{IJ} < 0\right\} = 1$. Then since player 1 is playing optimally,

$$\bar{v}(\phi_1, \phi_2) \leq E\bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}) .$$

From the monotonicity of $\bar{v}(\phi_1 + \varepsilon, \phi_2 - \varepsilon)$,

$$\bar{v}(\phi_1, \phi_2) \geqq \bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}) .$$

Combining,

$$\bar{v}(\phi_1, \phi_2) = \bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}) ,$$

or weaker, from monotonicity again,

$$\bar{v}(\phi_1, \phi_2) = \bar{v}(\phi_1 - 1, \phi_2 + 1) .$$

If $(\phi_1 - 1, \phi_2 + 1) > (0, 0)$, this implies that an optimal strategy I for

the first player for the game $\left(\bar{v}(\phi_1 + \Gamma_{ij} - 1, \phi_2 - \Gamma_{ij} + 1)\right)$ is an optimal strategy for $\left(\bar{v}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})\right)$, since by using it against any J, the first player insures himself

$$\begin{aligned} E\bar{v}(\phi_1 + \Gamma_{IJ}, \phi_2 - \Gamma_{IJ}) &\geqq E\bar{v}(\phi_1 + \Gamma_{IJ} - 1, \phi_2 - \Gamma_{IJ} + 1) \\ &\geqq \bar{v}(\phi_1 - 1, \phi_2 + 1) \\ &= \bar{v}(\phi_1, \phi_2) \ . \end{aligned}$$

Thus for a fortune division $(\phi_1 - 1, \phi_2 + 1) > (0, 0)$, and by induction for a fortune division, $(\phi_1 - n, \phi_2 + n) > (0, 0)$, for all optimal strategies, I, there is a J such that $\Gamma_{IJ} < 0$. But eventually, perhaps for $n = 0$,

$$(\phi_1 - n, \phi_2 + n) > (0, 0)$$

while $\phi_1 \leqq n + 1$. Therefore, for an optimal I and some J,

$$\begin{aligned} 0 < \delta \leqq v(\phi_1 - n, \phi_2 + n) &\leqq E\bar{v}(\phi_1 - n + \Gamma_{IJ}, \phi_2 + n - \Gamma_{IJ}) \\ &\leqq \bar{v}(\phi_1 - n - 1, \phi_2 + n + 1) \\ &= 0 \ , \end{aligned}$$

which is the contradiction we were looking for.

We have now proved that for $(\phi_1, \phi_2) > 0$, there is an optimal I such that for all J, $\Pr\left\{\Gamma_{IJ} > 0\right\} > 0$. For each $(\phi_1, \phi_2) > (0, 0)$, fix such an I. Call it $I(\phi_1, \phi_2)$. From the compactness of the second player's set of strategies and the fact that $\Pr\left\{\Gamma_{IJ} > 0\right\}$ is a continuous

function of his strategy, $\Pr\left\{\Gamma_{IJ} > 0\right\} \geqq \rho(\phi_1, \phi_2) > 0$. Define $\sigma(\phi_1, \phi_2) = \min_k \rho(\phi_1 + k, \phi_2 - k) > 0$, where k is an arbitrary positive, zero, or negative integer such that $(\phi_1 + k, \phi_2 - k) > (0, 0)$.

Let now player 1 use the conditionally optimal strategy which consists of playing $I(\phi_1, \phi_2)$ when the fortune distribution is (ϕ_1, ϕ_2). Let $Q^{(n)}$ be the probability that one or the other player's fortune is exhausted on or before the n-th play. Then, where $\sigma = \sigma(f_1, f_2)$,

$$Q^{([f_1+f_2]+1)} \geqq 0$$

$$Q^{(n+[f_1+f_2]+1)} \geqq Q^{(n)} + (1 - Q^{(n)})\sigma^{[f_1+f_2]+1} .$$

By induction,

$$Q^{(N\{[f_1+f_2]+1\})} \geqq 1 - \left(1 - \sigma^{[f_1+f_2]+1}\right)^{N-1} .$$

Hence $Q^{(N)} \longrightarrow 1$ as $N \longrightarrow \infty$, which is the lemma.

Let $\underline{\Omega}^{(n)}(f_1, f_2)$ be the game in which the two players repeat Γ n-times, or until one of the players has a non-positive fortune if this occurs first, and the money payoff is (1, 0) if player 2 ends with a non-positive fortune, (0, 1) otherwise. $\underline{\Omega}^{(n)}(f_1, f_2)$ is a constant-sum two-person game with value $\left(\underline{v}^{(n)}(f_1, f_2), 1 - \underline{v}^{(n)}(f_1, f_2)\right)$. Obviously

$$\underline{v}^{(n)}(f_1, f_2) \leqq \bar{v}^{(n)}(f_1, f_2) , \tag{5}$$

since any strategy for player 1 in $\underline{\Omega}^{(n)}(f_1, f_2)$, will insure him as much money in $\bar{\Omega}^{(n)}(f_1, f_2)$. We therefore conclude, by the same reasoning as

earlier,

(A) $\underline{v}^{(n)}(f_1, f_2) \longrightarrow \underline{v}(f_1, f_2) \in [0, 1 - \delta']$ if $(f_1, f_2) > (0, 0)$

$= 0$ if $f_1 \leqq 0$

$= 1$ if $f_2 \leqq 0$

where $\delta' > 0$.

(B) If $(f_1, f_2) > (0, 0)$,

$$\underline{v}(f_1, f_2) = G\big(\underline{v}(f_1 + \Gamma_{ij}, f_2 - \Gamma_{ij})\big) .$$

(C) If $\varepsilon \geqq 0$,

$$\underline{v}(f_1 + \varepsilon, f_2 - \varepsilon) \geqq \underline{v}(f_1, f_2) .$$

In addition

(D) $$\underline{v}(f_1, f_2) \leq \bar{v}(f_1, f_2) .$$

Definition: A strategy for player 2 is called conditionally optimal if the conditional distribution of his strategy at any play of Γ, given the course of the game up to that play, is an optimal strategy for the game $\big(\bar{v}(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})\big)$ where (ϕ_1, ϕ_2) is the fortune distribution immediately before the play in question.

From Lemmas $\bar{1}$ and $\bar{2}$, and the analogous Lemmas $\underline{1}$ and $\underline{2}$ which we do not write down, we conclude that each player has a conditionally optimal strategy which insures that play ends by the n-th play with probability

tending to 1 as n tends to ∞, uniformly in the opponent's strategy. The first player's strategy insures him $\bar{v}(f_1, f_2)$ on the average and the second player's strategy insures him $1 - \underline{v}(f_1, f_2) \geqq 1 - \bar{v}(f_1, f_2)$, on the average. Since together the players can win no more than 1, we get

$$\begin{aligned} 1 &\geqq \bar{v}(f_1, f_2) + (1 - \underline{v}(f_1, f_2)) \\ &\geqq \bar{v}(f_1, f_2) + (1 - \bar{v}(f_1, f_2)) \\ &= 1 . \end{aligned}$$

This means $\underline{v}(f_1, f_2) = \bar{v}(f_1, f_2) =$ (say) $v(f_1, f_2)$, and $\Omega(\phi_1, \phi_2)$ is inessential with the solution $\{(v(f_1, f_2), 1 - v(f_1, f_2))\}$.

v can be characterized as the unique solution of

$$\begin{aligned} 0 \leqq v(\phi_1, \phi_2) &= G(v(\phi_1 + \Gamma_{ij}, \phi_2 - \Gamma_{ij})) \leqq 1 \\ &\qquad \text{if } (f_1, f_2) > (0, 0) \\ &= 0 \quad \text{if } f_1 \leqq 0 \\ &= 1 \quad \text{if } f_2 \leqq 0 . \end{aligned}$$

For if v^* is a solution,

$$\underline{v}^{(0)}(\phi_1, \phi_2) \leqq v^*(\phi_1, \phi_2) \leqq \bar{v}^{(0)}(\phi_1, \phi_2)$$

by definition, and so by induction, using ($\bar{A}$), ($\bar{B}$), ($\underline{A}$), and ($\underline{B}$),

$$\underline{v}^{(n)}(\phi_1, \phi_2) \leqq v^*(\phi_1, \phi_2) \leqq \bar{v}^{(n)}(\phi_1, \phi_2) .$$

Hence

$$v(\phi_1, \phi_2) = \underline{v}(\phi_1, \phi_2) \leqq v^*(\phi_1, \phi_2) \leqq \bar{v}(\phi_1, \phi_2) = v(\phi_1, \phi_2) ,$$

giving

$$v(\phi_1, \phi_2) = v^*(\phi_1, \phi_2) ,$$

as was to be proved.

We thus have

Theorem 1: $\Omega(f_1, f_2)$ *is inessential with the solution* $\{(v(f_1, f_2), 1 - v(f_1, f_2))\}$ *where* v *is the unique solution in* $\{(\phi_1, \phi_2) | \phi_1 > 0$ *or* $\phi_2 > 0\}$ *of*

$$\begin{aligned} 0 \leqq v(\phi_1, \phi_2) &= G(v(\phi_1 + r_{ij}, \phi_2 - r_{ij})) \leqq 1 \qquad \text{if } (\phi_1, \phi_2) > (0, 0) \\ &= 0 \qquad \text{if } \phi_1 \leqq 0 \\ &= 1 \qquad \text{if } \phi_2 \leqq 0 . \end{aligned}$$

Each player has a conditionally optimal strategy which is optimal and which insures that play ends by the n-th play with probability tending to one uniformly in the opponent's strategies.

Let us turn now to the problem of effectively computing an ϵ-optimal strategy for $\Omega(f_1, f_2)$. This is easy if we are not interested in efficiency. Namely, we need only find an n such that $\overline{v}^{(n)}(f_1, f_2) - \underline{v}^{(n)}(f_1, f_2) \leqq \epsilon - \delta$ where $\delta > 0$. Then a δ-optimal strategy for the first player for

$\Omega^{(n)}(f_1, f_2)$ provides an ε-optimal strategy for him for $\Omega(f_1, f_2)$. Namely, he can use the strategy on the first n moves of $\Omega(f_1, f_2)$ and act arbitrarily thereafter. Similarly, a δ-optimal strategy for the second player for $\bar{\Omega}^{(n)}(f_1, f_2)$ provides an ε-optimal strategy for him for $\Omega(f_1, f_2)$.

If $\max_{i,j}|\Gamma_{ij}|$ is small enough compared to f_1 and f_2, there is another class of interesting ε-optimal strategies. Repeatedly playing an optimal strategy for Γ is an ε-optimal strategy for Ω. More precisely, let us remove the restriction that each Γ_{ij} be a non-zero integer. Let us require instead, say, that $G(\Gamma) \geqq 0$ and that for some optimal strategy I, $\Pr\left\{\Gamma_{Ij} > 0\right\} > 0$ for all j. If $G(\Gamma) = 0$, we require in addition that for some optimal J, $\Pr\left\{\Gamma_{iJ} < 0\right\} > 0$ for all i. Define $\alpha = G(\Gamma)$, $\beta = \min_j \Pr\left\{\Gamma_{Ij} > 0\right\}$, $\gamma = \max_{i,j} |\Gamma_{ij}|$.

We assume that both f_1 and f_2 are positive and define $f = f_1 + f_2$. Define for $\alpha = 0$

$$
\begin{aligned}
p_0(\phi_1) &= \frac{1}{f+\gamma}\,\phi_1 && \text{if} \quad 0 < \phi_1 < f \\
&= 0 && \text{if} \quad \phi_1 \leqq 0 \\
&= 1 && \text{if} \quad \phi_1 \geqq f
\end{aligned}
$$

and for $\alpha > 0$

$$p_\alpha(\phi_1) = \frac{1 - \exp\left\{-\frac{\alpha}{\gamma^2}\phi_1\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2}(f + \gamma)\right\}} \quad \text{if} \quad 0 < \phi_1 < f$$

$$= 0 \qquad \text{if} \quad \phi_1 \leqq 0$$

$$= 1 \qquad \text{if} \quad \phi_1 \geqq f \ .$$

Lemma $\overline{3}$: <u>If player 1 plays I repeatedly, then he can expect at least $p_\alpha(f_1)$ in payoff</u>. (I is any optimal strategy for Γ satisfying $\Pr(\Gamma_{Ij} > 0) > 0$.)

Proof: Since $\beta > 0$, by the method of proof of Lemma $\overline{2}$, it follows that if player 1 plays I repeatedly, the probability that the game ends by the n-th play tends to one as n tends to ∞. Hence, in order to prove Lemma $\overline{3}$, it is sufficient to show that for all N,

$$Ep_\alpha(F_1^N) \geqq p_\alpha(f_1) \ .$$

By induction, this would follow from

$$E\left\{p_\alpha(F_1^{N+1}) \mid F_1^N\right\} \geqq p_\alpha(F_1^N) \ .$$

We prove the latter.

Suppose first that $\alpha = 0$. If $0 < F_1^N < f$, then for all (i, j), since $\Gamma_{ij} \leqq \gamma$,

$$p_0(F_1^N + \Gamma_{ij}) \geqq \frac{1}{f + \gamma}(F_1^N + \Gamma_{ij}) \ .$$

Hence, if $0 < F_1^N < f$,

$$E\left\{p_0(F_1^{N+1})\,|F_1^N\right\} \geqq \min_j \, Ep_0(F_1^N + \Gamma_{Ij})$$

$$\geqq \frac{1}{f+\gamma} \min_j \, E(F_1^N + \Gamma_{Ij})$$

$$\geqq \frac{1}{f+\gamma} F_1^N$$

$$= p_0(F_1^N) \; .$$

Since if $F_1^N \leqq 0$ or $F_1^N \geqq f$ our proposition is trivial, we have disposed of the case $\alpha = 0$.

Suppose now that $\alpha > 0$. Again we need only consider $0 < F_1^N < f$. Then

$$p_\alpha(F_1^N + \Gamma_{ij}) \geqq \frac{1 - \exp\left\{-\frac{\alpha}{\gamma^2}(F_1^N + \Gamma_{ij})\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2}(f+\gamma)\right\}} \; .$$

Hence,

$$E\left\{p_\alpha(F_1^{N+1})\,|F_1^N\right\} \geqq \min_j \, Ep_\alpha(F_1^N + \Gamma_{Ij})$$

$$\geqq \min_j \frac{1 - E\exp\left\{-\frac{\alpha}{\gamma^2}(F_1^N + \Gamma_{Ij})\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2}(f+\gamma)\right\}}$$

$$\geqq \frac{1 - M\exp\left\{-\frac{\alpha}{\gamma^2}F_1^N\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2}(f+\gamma)\right\}}$$

where

$$M = \max_j E \exp\left\{-\frac{\alpha}{\gamma^2}\Gamma_{Ij}\right\}$$

$$\leqq \max_j \left\{1 - \frac{\alpha}{\gamma^2} E\,\Gamma_{Ij} + (e-2)\left(\frac{\alpha}{\gamma}\right)^2\right\}$$

$$\leqq \left\{1 - \frac{\alpha^2}{\gamma^2} + (e-2)\frac{\alpha^2}{\gamma^2}\right\}$$

$$< 1 .$$

Hence

$$E\left\{p_\alpha(F_1^{N+1})\,|F_1^N\right\} \geqq \frac{1 - \exp\left\{-\frac{\alpha}{\gamma^2}F_1^N\right\}}{1 - \exp\left\{-\frac{\alpha}{\gamma^2}(f+\gamma)\right\}}$$

$$= p_\alpha(F_1^N) ,$$

as was to be proved.

By symmetry, if $\alpha = 0$ we conclude

Lemma $\underline{3}$. _If_ $\alpha = 0$, _and if player 2 plays J repeatedly, then he can expect at least_ $p_0(f_2)$ _in payoff_. (J is any optimal strategy for Γ satisfying Pr $(\Gamma_{iJ} < 0) > 0$.)

If $\alpha = 0$, Lemmas $\overline{3}$ and $\underline{3}$ give us, whenever $\Omega(f_1, f_2)$ is inessential with solution $\left\{(v(f_1, f_2), 1 - v(f_1, f_2)\right\}$,

$$p_0(f_1) \leqq v(f_1, f_2) \leqq 1 - p_0(f_2) = p_0(f_1) + \frac{\gamma}{f+\gamma} .$$

Thus repeating I is $(\frac{\gamma}{f + \gamma})$–optimal for player 1, and repeating J is $(\frac{\gamma}{f + \gamma})$–optimal for player 2. If $\alpha > 0$, Lemma $\overline{3}$ gives us whenever $\Omega(f_1, f_2)$ is inessential with solution $\{(v(f_1, f_2), 1 - v(f_1, f_2))\}$,

$$1 - \exp\left\{- \frac{\alpha}{\gamma^2} f_1\right\} \leqq p_\alpha(f_1) \leqq v(f_1, f_2) \leqq 1 .$$

Thus repeating I is $\exp\left\{- \frac{\alpha}{\gamma^2} f_1\right\}$–optimal for player 1 and any strategy is $\exp\left\{- \frac{\alpha}{\gamma^2} f_1\right\}$–optimal for player 2.

What if, instead of repeating I, player 1 repeated a δ–optimal $I\delta$, where δ is the smallest number for which $I\delta$ is δ–optimal? If $\alpha > \delta$ no great harm is done, since it can be verified by precisely the proof given above that this is an $\exp\left\{- \frac{\alpha - \delta}{\gamma^2} f_1\right\}$–optimal strategy for player 1. If, however, $\alpha < \delta$, player 2 could expect at least $1 - \exp\left\{- \frac{\delta - \alpha}{\gamma^2} f_2\right\}$ in payoff. When $\frac{\delta - \alpha}{\gamma^2} f_2$ is large, this payoff is close to one, so $I\delta$ is not a good strategy. Thus if $\alpha = 0$, no matter how small γ is, it is not enough to repeat a δ–optimal strategy for sufficiently small δ. On the other hand, suppose that (I_n) is a sequence of strategies for player 1 whose n–th member is δ^n–optimal for Γ and satisfies

$$\min_j \Pr\left\{\Gamma_{I_n j} > 0\right\} \geqq \beta' > 0,$$

where β' does not depend on n. Then

Lemma 4: *If* $\alpha = 0$ *and player* 1 *plays* I_n *at the n-th stage, then he can expect at least* $p_\alpha(f_1) - \frac{\delta}{(1-\delta)(f+\gamma)}$ *in payoff.*

Proof: The proof is almost identical with that of Lemma 3 where instead of proving

$$Ep_\alpha(F_1^N) \geqq p_\alpha(f_1)$$

one proves

$$Ep_\alpha(F_1^N) \geqq p_\alpha(f_1) - \frac{\delta + \cdots + \delta^{N-1}}{f+\gamma} .$$

It is an easy step (left to the reader) now to

Theorem 2. *If* $G(\Gamma) = \alpha > \delta$ *and* $\Omega(f_1, f_2)$ *is inessential, repeating a strategy which is* δ*-optimal for* Γ *is* $\exp\left\{-\frac{\alpha-\delta}{\gamma^2} f_2\right\}$*-optimal for* $\Omega(f_1, f_2)$. *Let* $G(\Gamma) = 0$, *and let* (I_n) *be a sequence of strategies for player* 1 *whose n-th member is* δ^n*-optimal for* Γ *and satisfies*

$$\min_j \Pr\left\{\Gamma_{I_n j} > 0\right\} \geqq \beta' > 0 ,$$

where β' *does not depend on* n. *Then playing* I_n *at the n-th stage is a* $\left(\frac{\gamma}{f+\gamma} + \frac{2\delta}{(1-\delta)(f+\gamma)}\right)$*-optimal strategy for player* 2.

The reader will observe that when each $\Gamma_{ij} \neq 0$, say $|\Gamma_{ij}| \geqq C$, we automatically have for a δ^n-optimal I_n, when δ is sufficiently small,

$$\min_j \Pr\left\{\Gamma_{I_n j} > 0\right\} \geqq \frac{C - \delta^n}{C + \gamma} \geqq \frac{C - \delta}{C + \gamma} > 0 \ .$$

In closing, we wish to point out that the method of proof leading to Theorem 1 is trivially sufficient to handle the following generalized game of survival in which the result of a play is a random state instead of a definite number. However, the method is apparently insufficient to handle more than a finite number of possible states, or the possibility of "zeros." A finite set Σ with two distinguished points σ_1 and σ_2 is given. Σ is partially ordered by <, which satisfies for some fixed n and all $\left\{x_i \,|\, 1 \leqq i \leqq n\right\}$,

$$x_1 < x_2 < \cdots < x_{n-1} < x_n \longrightarrow x_1 = \sigma_2,\ x_n = \sigma_1 \ .$$

For each $x \in \Sigma$, there is a set of random variables on Σ, $\left\{Y_{ij}(x) \,|\, 1 \leqq i \leqq i_0,\ 1 \leqq j \leqq j_0\right\}$ such that for all i and j $Y_{ij}(\sigma_1) = \sigma_1$, $Y_{ij}(\sigma_2) = \sigma_2$, and for $x \neq \sigma_1, \sigma_2$

$$\Pr\left\{Y_{ij}(x) < x\right\} = 0 \longrightarrow \Pr\left\{x < Y_{ij}(x)\right\} = 1$$

$$\Pr\left\{x < Y_{ij}(x)\right\} = 0 \longrightarrow \Pr\left\{Y_{ij}(x) < x\right\} = 1 \ .$$

In addition, for $x \neq \sigma_1, \sigma_2$, for each i, there is a j such that

$$\Pr \left\{ Y_{ij}(x) < x \right\} > 0 ,$$

and for each j, there is an i such that

$$\Pr \left\{ x < Y_{ij}(x) \right\} > 0 .$$

Define $\prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x)$ by induction by

$$\prod_{n=1}^{M+1} Y_{i_n j_n}^{(n)}(x) = Y_{i_{M+1} j_{M+1}}^{M+1}\left(\prod_{n=1}^{M} Y_{i_n j_n}^{(n)}(x)\right) ,$$

where $\left\{\left(Y_{ij}^{(n)}(x) \mid 1 \leqq i \leqq i_o,\ 1 \leqq j \leqq j_o,\ x \in \sum\right)\right\}$ is a set of independent random variables, each distributed like $(Y_{ij}(x))$. Then we finally require that $x < x'$ implies that for all N

$$\Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x') = \sigma_1 \right\} \geqq \Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x) = \sigma_1 \right\} ,$$

$$\Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x') = \sigma_2 \right\} \leqq \Pr \left\{ \prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x) = \sigma_2 \right\} .$$

All that we have said about $\left\{\Omega(f_1, f_2)\right\}$ up to Theorem 1, trivially modified, applies to the games $\left\{\Omega(x)\right\}$ in which two players repeatedly and simultaneously choose integers i_n and j_n

at each time n, until $\prod_{n=1}^{N} Y_{i_n j_n}^{(n)}(x) = \sigma_1$ or σ_2, or *ad infinitum* if this never occurs. The payoff is (1, 0) if the game ends in the state σ_1, and (0, 1) if the game ends in the state σ_2. If the game goes on indefinitely, then the payoff is $(\alpha(C), \beta(C))$ where $(\alpha(C), \beta(C)) \leqq (1, 1)$ and $\alpha(C) + \beta(C) \leqq 1$, where $(\alpha(C), \beta(C))$ can depend on the course of the game, C.

Similarly, Theorem 2 can be generalized by the use of expected values to the situation where $\sum$ is a set of reals and for $\sigma_1 > x > \sigma_2$

$$
\begin{aligned}
Y_{ij}(x) &= x + a_{ij} && \text{if} \quad \sigma_2 < x + a_{ij} < \sigma_1 \\
&= \sigma_1 && \text{if} \quad \sigma_1 \leqq x + a_{ij} \\
&= \sigma_2 && \text{if} \quad \sigma_2 \geqq x + a_{ij} ,
\end{aligned}
$$

where a_{ij} is a real-valued random variable whose distribution depends on (i, j).